Polynomial functions

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Polynomial functions

$f(z) = \sum_{n=0}^{d} a_n z^n$ By Theorem, $f'(z) = \sum_{n=1}^{d} n a_n z^{n-1}$. Entire function.

Analytic $\forall z \in \mathbb{C}$.

$p(z) = z^2 = (x^2 - y^2) + 2ixy$ $x = \frac{z + \bar{z}}{2}$

$y = \frac{z - \bar{z}}{2i}$

A little bit about polynomials:

Fundamental Theorem of Algebra (will prove it later):

$p(z)$ - polynomial of degree $d > 1 \Rightarrow p$ has a root (or zero) z_1,

$p(z_1) = 0$.

Divide p by $(z - z_1)$: $p(z) = q(z)(z - z_1)$ (remainder is a constant, so it is $p(z_1) = 0$)

q - has degree $d-1$, so also has a root.

So $p(z) = a_d (z - z_1) \dots (z - z_d)$ - by induction.

Some of the roots can be the same! Multiplicity or order of

z_0 as a root: $\#\{j : z_j = z_0\}$.

I.e. $p(z) = (z - z_0)^h q(z)$, $q(z_0) \neq 0$. h - order.

Equivalently: $p(z_0) = p'(z_0) = \dots = p^{(h-1)}(z_0) = 0$, $p^{(h)}(z_0) \neq 0$.

Pf. $p'(z) = h(z - z_0)^{h-1} q(z) + (z - z_0)^h q'(z) =$

$$(z - z_0)^{h-1} \underbrace{(h q(z) + (z - z_0) q'(z))}_{r(z)}$$

$r(z_0) = h q(z_0) \neq 0$, so has order $h-1$ for p' ∎

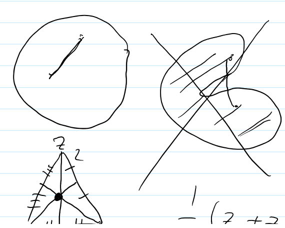
Carl Friedrich Gauss

Theorem (Gauss)

Let $z_1, \dots, z_d$ be zeroes of a polynomial p.
Then all the zeroes of $p'(z)$ lie inside the convex hull of $\{z_1, \dots, z_d\}$.

Def. Convex hull of $\{z_1, \dots, z_d\}$ in

$$H = \{w \in \mathbb{C} : \exists m_1, \dots, m_d : m_j \geq 0, \sum_{j=1}^{d} m_j = 1, w = \sum_{j=1}^{d} m_j z_j\}.$$



Geometric meaning: the smallest convex set

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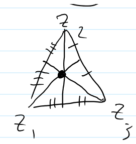
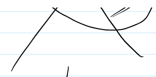
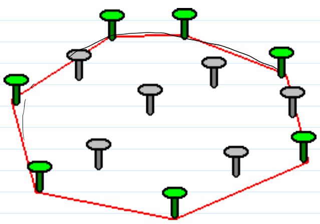
$w = \sum_{j=1}^{d} m_j z_j \}.$

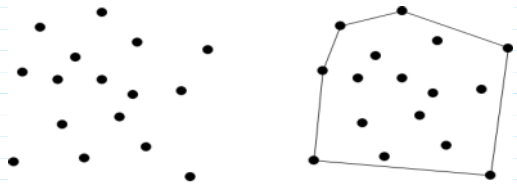
Geometric meaning: the smallest convex set containing $\{z_1, \dots, z_d\}$.





$\frac{1}{3}(z_1 + z_2 + z_3) = \frac{1}{3} z_1 + \frac{1}{3} z_2 + \frac{1}{3} z_3$





(a) (b)

$(p(z) - c)' = p'(z)$

$p(z) = c$

Corollary. $\forall c \in \mathbb{C}$ the roots of $p'(z)$ lie in the convex hull of the roots $p(z) = c$.

Proof of Theorem

$$\frac{p'(z)}{p(z)} = \sum_{j=1}^{d} \frac{1}{z - z_j} = \sum \frac{\overline{z - z_j}}{|z - z_j|^2}$$

If z_0 – multiple root of p – nothing to prove!

If z_0 is not a root of p, but a root of p' then $\overline{\frac{p'(z_0)}{p(z_0)}} = 0.$

So $\sum \frac{z_0 - z_j}{|z_0 - z_j|^2} = 0$. Or $z_0 = \sum m_j z_j$, where

$$m_j = \frac{|z_0 - z_j|^{-2}}{\sum_{j=1}^{d} |z_0 - z_j|^{-2}}, \quad \sum m_j = 1, \quad m_j \geq 0 \ \blacksquare$$

Brook Taylor

Taylor polynomial.

Thm. Let $p(z)$ be a polynomial of degree d, $z_0 \in \mathbb{C}$. Then $p(z) = \sum_{k=0}^{d} \frac{p^{(k)}(z_0)}{k!} (z - z_0)^k$

Proof Induction on d. $d = 0$ $p(z) = C = p(z_0)$.

Step: $\frac{p(z) - p(z_0)}{z - z_0} =: q(z)$ – polynomial.

$p(z) = p(z_0) + (z - z_0) q(z)$, $\quad p'(z) = q(z) + (z - z_0) q'(z)$

$p''(z) = 2 q'(z) + (z - z_0) q''(z)$

$p^{(k)}(z) = k q^{(k-1)}(z) + (z - z_0) q^{(k)}(z)$

$p^{(k)}(z_0) = k\, q^{(k-1)}(z_0)$.

$$p(z) = p(z_0) + (z-z_0) \sum_{k=0}^{d-1} \frac{q^{(k-1)}(z_0)}{(k-1)!}(z-z_0)^{k-1} = p(z_0) + \sum_{k=1}^{d} \frac{p^{(k)}(z_0)}{k!}(z-z_0)^k$$

induction!